

QUASI ANALYTICITY RELATED TO SETS OF POSITIVE MEASURE
FOR FUNCTIONS OF A COMPLEX VARIABLE

BY

FLOYD FRANKLIN HELTON

A.B., Westminster College, 1935
M.A., University of Missouri, 1939

AN ABSTRACT OF A THESIS

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY IN MATHEMATICS
IN THE GRADUATE SCHOOL OF THE UNIVERSITY OF
ILLINOIS, 1946

URBANA, ILLINOIS
1946

QUASI ANALYTICITY RELATED TO SETS OF POSITIVE MEASURE
FOR FUNCTIONS OF A COMPLEX VARIABLE

BY
FLOYD FRANKLIN HELTON

A.B., Westminster College, 1935
M.A., University of Missouri, 1939

AN ABSTRACT OF A THESIS

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY IN MATHEMATICS
IN THE GRADUATE SCHOOL OF THE UNIVERSITY OF
ILLINOIS, 1946

URBANA, ILLINOIS
1946



QUASI ANALYTICITY RELATED TO SETS OF POSITIVE MEASURE
FOR FUNCTIONS OF A COMPLEX VARIABLE

1. Introduction

Problems of representation and uniqueness play a large role in the theory of general monogenic functions of a complex variable developed by W. J. Trjitzinsky. Corresponding to the Cauchy integral formulas for analytic functions and to certain analogous contour integral formulas for Borel monogenic functions, at the basis of this function theory are certain two-dimensional Lebesgue integrals. This paper is specifically concerned with functions

$$(1.1) \quad f(w) = \iint_{CE} \frac{q(x,y) \, dx \, dy}{z - w}, \quad z = x + iy,$$

$w \in E$, $mE > 0$, where E is a closed subset of a bounded domain K , and $q(x,y)$ is a function continuous on $\bar{K}$ and vanishing on E .

For such functions $f(w)$, quasi-analyticity in the sense of unique determination by values of the function and of the derivatives of all orders at a point, and in various other senses, have been studied by Trjitzinsky and others. It is the purpose of this paper to develop conditions for quasi-analyticity of $f(w)$ in the sense of unique determination by values on a suitable subset of E having positive Lebesgue planar measure. The extent of the domain in which uniqueness is secured will be related (1) to the 'density' of $q(x,y)$ on K , and (2) to the 'rarefaction' of CE .

Fundamental to these results is a theorem by Trjitzinsky concerning limits of sequences of analytic functions. Because of its complexity and length, it will not be quoted here, but the reader is referred to W. J. Trjitzinsky, Problems of Representation and Uniqueness for Functions of a Complex Variable, Acta Mathematica, vol. 78, (1946), pp. 97-192 and in particular to theorem 5.2, pages 168-9. In the sequel it will be referred to as (T).

2. Uniqueness on the Basis of Density of $q(x,y)$ on K .

Consider the function $f(w)$ defined in (1.1). Denote by E_v the set of points of K whose distance from E is less than $1/v$. It may be shown that $f(w)$ is analytic at interior points of E (actually, however, E may be without interior points) and that

$$(2.1) \quad f_n(w) = \iint_{CE_n} \frac{q(x,y) \, dx \, dy}{z - w}, \quad z = x + iy,$$

$w \in E_n$, is analytic in the open set E_n .

Noting that $CE - CE_n = \sum_{v=n}^{\infty} (CE_{v-1} - CE_v)$ and that $|q(x,y)| \leq b(\rho)$, where $\rho(z,E)$ is the distance of the point (x,y) from the frontier of CE , and where $b(\rho)$ may be taken monotonically decreasing with ρ , we obtain

$$\begin{aligned} (2.2) \quad |f(w) - f_n(w)| &\leq \iint_{CE - CE_n} \left| \frac{q(x,y)}{z - w} \right| \, dx \, dy \\ &\leq \sum_{v=n}^{\infty} b(\rho) \, (v+1) \, m(CE_{v+1} - CE_v) \\ &\leq o \sum_{v=n}^{\infty} v \, b(1/v). \end{aligned}$$

Suitable conditions on the rapidity with which $b(\rho) \rightarrow 0$ will secure the uniform convergence in E of the sequence of analytic functions $\{f_n(w)\}$ to $f(w)$.

Preliminary to subjecting $f(w)$ to the conditions of (T), we construct a function $k(n)$, continuous, monotone increasing, and defined for all $n \geq 1$, such that $k(n) = k_n$ for integral values of n , where $\{k_n\}$ is a monotone increasing sequence of constants defined in (T). Further let τ ($0 < \tau < 1$), be defined as in (T).

Suppose now that $b(\rho) \rightarrow 0$ so rapidly with ρ that

$$(A) \quad \sum_{v=1}^{\infty} v b(1/v) \leq c^n n b(1/n)$$

and

$$(B) \quad b(\rho) \leq \rho \psi(\rho) \tau^{k(1/\rho)}$$

where $\psi(\rho)$ is any positive function tending to zero with ρ ,
Then for $\rho = 1/n$, ($n = 1, 2, \dots$),

$$(2.3) \quad n b(1/n) \leq \psi(1/n) \tau^{k_n}.$$

Applying these conditions to (2.2), we obtain

$$(2.4) \quad |f(w) - f_n(w)| \leq \phi(n) \tau^{k_n},$$

where $\phi(n) \rightarrow 0$ with $1/n$. Therefore, we may invoke (T), and state the

THEOREM. Let

$$f(w) = \iint_{CE} \frac{q(x,y) \, dx \, dy}{z - w},$$

$w \in E$ and $z = x + iy$, where E ($mE > 0$), is a closed subset of a bounded domain K and $q(x,y)$ is a function continuous on $\bar{K}$ and

vanishing on E (in consequence of which $|q(x,y)| \leq b(\rho)$, where ρ is the distance of (x,y) from the frontier of CE, and where $b(\rho)$ may be taken monotone decreasing). Suppose $E_0 \subset E$ such that we have $\frac{1}{2} mE - q < mE_0 < \frac{1}{2} mE$, with q defined in (T). Consider the consequences of the vanishing of $f(w)$ in E_0 .

If $b(\rho) \rightarrow 0$ so rapidly with ρ that

$$(A) \quad \sum_{v=1}^{\infty} v b(1/v) \leq c'' n b(1/n),$$

in which c'' is independent of n , and

$$(B) \quad b(\rho) \leq \rho \psi(\rho) \tau^{k(1/\rho)},$$

where $\psi(\rho)$, $k(1/\rho)$, and $b(\rho)$ are defined earlier, and τ is given in (T), then

$$(2.5) \quad f(w) = 0 \quad \text{in } E',$$

where $E_0 \subset E' \subset E$ and $mE' \geq \frac{1}{2} mE$.

If in (B), τ may be replaced by ω (as given in (T)), then

$$(2.6) \quad f(w) = 0 \quad \text{in } E_\omega,$$

where $E_0 \subset E_\omega \subset E$ and $mE_\omega \geq \frac{1}{2} mE + q - \epsilon$, $0 < \epsilon < q$.

Finally, if in (B), τ may be replaced by $\tau(\sigma)$, where $\tau(\sigma) > 0$ and $\tau(\sigma) \rightarrow 0$ monotonically with $1/\sigma$, then

$$(2.7) \quad f(w) = 0 \quad \text{in } E_\sigma,$$

where $E_0 \subset E_\sigma \subset E$ and $mE_\sigma \geq \frac{1}{2} mE + q$.

3. Uniqueness on the Basis of Rarefaction of CE.

Let G , $mG > 0$, be a closed subset of a bounded domain K and suppose that $CG = K - G$ be covered by a sequence of circular regions

$$(3.1) \quad G_1: |z - \alpha_1| < \sigma_1, \quad (\alpha_1 \in K, z \in CG, \sigma_1 \rightarrow 0).$$

Take $\rho = \lambda \sigma_i$, ($\lambda > 1$), and let D_1 be the part of K for which we have $|z - \alpha_i| \geq \rho$. Then $D = \sum_{i=1}^{\infty} D_1 \subset G$. Denote by Q_1 the part of CG interior to C_1 and such that $Q_1 Q_{1-i} = 0$ for $i = 2, 3, \dots$.

Then

$$(3.2) \quad CG = Q_1 + Q_2 + Q_3 + \dots$$

$$\text{Write } G_n = G + \sum_{i=n+1}^{\infty} Q_i, \quad (n = 1, 2, \dots).$$

$$\text{Then } K \supset G_1 \supset G_2 \supset G_3 \dots \longrightarrow G,$$

$$\text{and } CG_n = \sum_{i=1}^n Q_i.$$

Now let $q(x, y)$ be a function measurable and bounded on K , and hence integrable, and consider the functions defined by

$$(3.3) \quad f(w) = \iint_{CG} \frac{q(x, y) \, dx \, dy}{z - w} = \sum_{i=1}^{\infty} \iint_{Q_i} \frac{q(x, y) \, dx \, dy}{z - w}, \quad w \in G,$$

$$(3.4) \quad f_n(w) = \iint_{CG_n} \frac{q(x, y) \, dx \, dy}{z - w} = \sum_{i=1}^n \iint_{Q_i} \frac{q(x, y) \, dx \, dy}{z - w}, \quad w \in G_n.$$

By means of the reasoning employed in the previous section it may be shown that $f(w)$ and $f_n(w)$ are analytic at interior points of G and G_n , respectively, and hence analytic at interior points of $D \subset G \subset G_n$.

Now for $z \in Q_1$ and $w \in D \subset G$,

$$(3.5) \quad |z - w| \geq \rho - \sigma_i = \sigma_i (\lambda - 1).$$

From this and the fact that $|q(x, y)| \leq b$ and $mQ_1 \leq \pi \sigma_i^2$, we obtain

$$(3.6) \quad \begin{aligned} |f(w) - f_n(w)| &\leq \sum_{i=n+1}^{\infty} b \, mQ_i / \sigma_i (\lambda - 1) \\ &\leq c \sum_{i=n+1}^{\infty} \sigma_i, \quad \text{independent of } w. \end{aligned}$$

Impose on the σ_i the conditions

$$(C) \quad \sigma_{n+1} \leq \psi(n) \uparrow k_n$$

$$(D) \quad \sum_{i=n+1}^{\infty} \sigma_i \leq c' \sigma_{n+1}, \quad c' \text{ independent of } n,$$

where $\psi(n)$, $\uparrow$, and k_n are as previously defined. The condition (D) insures that $f_n(w) \rightarrow f(w)$ uniformly in $D \subset G$.

Now, as in the preceding section, we have

$$(3.7) \quad |f(w) - f_n(w)| \leq \delta(n) r^{k_n},$$

and with λ so chosen that $mD > 0$, we may state the Theorem for these conditions also.

4. Weakening of the Conditions.

The conditions developed above are somewhat stringent. Under suitable circumstances they may be relaxed. In particular, if $m(CE_{v+1} - CE_v) = m(E_v - E_{v+1}) = O(1/v^s)$, it can be shown that (A) and (B) may be replaced by the weaker conditions

$$(A') \quad \sum_{v=1}^{\infty} (1/v^{s-1}) b(1/v) \leq (c'/n^{s-1}) b(1/n)$$

and $(B') \quad b(\rho) \leq r^{k(1/\rho)}.$

That such Conditions actually obtain is shown by examples, and closed sets of positive measure and without interior points are exhibited for which $m(E_v - E_{v+1}) = O(1/v^s)$ for $1 < s \leq 2$.

5. Conclusion.

The theorems developed in this paper give conditions under which certain classes of functions important in the representation of general monogenic functions are quasi-analytic in the sense of unique determination of members of their class by their values on certain sets of positive measure. The conditions are related (1) to the density of $q(x, y)$ of functions (1.1) and (2) to the rarefaction of the set CE complementary to the domain of definition of the given functions.

BIBLIOGRAPHY

E. Borel, *Leçons sur les Fonctions Monogènes*, Gauthier-Villars, Paris, 1917.

W.J. Trjitzinsky, *Théorie des Fonctions d'une Variable Complexe Définies sur des Ensembles Généraux*, Ann. Ec. Norm., (3), LV., Fasc. 2, pp. 119-91.

W.J. Trjitzinsky, *Some General Developments in the Theory of Functions of a Complex Variable*, Acta Mathematica, vol. 70, (1938), p. 63-163.

W.J. Trjitzinsky, *Problems of Representation and Uniqueness for Functions of a Complex Variable*, Acta Mathematica, vol. 78, (1946), p. 97-192.

C. de la Vallée Poussin, *Intégrales de Lebesgue*, Gauthier-Villars, Paris, 1934.

S. Saks, *Theory of the Integral*, Warszawa-Lwow, 1937.

Titchmarsh, *The Theory of Functions*, Second Edition, Oxford, 1939.

VITA

Floyd Franklin Helton was born September 6, 1912, at Council Bluffs, Iowa, and his elementary education was received in the public schools of Missouri. He graduated from the high school of the School of the Ozarks, Point Lookout, Missouri, in 1931, and received the A.B. degree from Westminster College, Fulton, Missouri, in 1935. The next three years were spent in secondary school teaching: Principal, Tebbetts, Missouri, High School, 1935-6, and Teacher of Mathematics and Science, Koshkonong, Missouri, High School, 1936-8. He did graduate work in Mathematics at the University of Missouri during the summers of 1936-9, inclusive, and during the academic year, 1938-9, and received the degree of Master of Arts, with major in Mathematics, from the University of Missouri, August, 1939. He was Instructor in Mathematics at Central College, Fayette, Missouri, 1939-41, with the intervening summer of 1940 spent in graduate study at the University of Wisconsin. From 1941 until 1944, he held a graduate assistantship and did graduate work in the Department of Mathematics of the University of Illinois. The summers of 1945-6 were also spent in graduate work there. Since November, 1944, he has been Associate Professor of Mathematics, and Acting Head of the Department of Mathematics, Central College, Fayette, Missouri. He is a member of Pi Mu Epsilon, the Mathematical Association of America, the American Mathematical Society, Sigma Xi, the American Association for the Advancement of Science, and a charter member of the Missouri Academy of Science.

